RESEARCH STATEMENT

YU ZHAO

I am a geometric representation theorist, who aims to build bridges among geometry, physics, and representation theory.

Originated from the categorical representation theory, currently I am focusing on introducing birational geometry methods to derived schemes/stacks, as observing the surprising and striking relation between many mathematical phenomenons arising from varying fields and seeming irrelevant, including

(1) the enumerative geometry (like the wall-crossing and virtual localization formulas),
(2) the derived category of coherent sheaves of algebraic varieties (like the semi-orthogonal decomposition of blow-up/stacky-blow-up of smooth varieties),
(3) the categorical representation theory (like the categorification of quantum loop/toroidal/shifted algebras)

with a natural generalization of a classical vanishing theorem from birational geometry, after considering the derived schemes. In this research statement, we will explain this observation and prospect the potential developments in the future through concrete examples.

1. Research Progress

1.1. **The generalized vanishing theorem for blow-ups.** The study of blowing-up in algebraic varieties has a rather long history, which can be traced back to the Italian school in the 19th century. Here we take the simplest example where Z is the affine space $\mathbb{A}^m$ and X is the origin. Then the blow-up variety $Bl_X Z$ is

$$Bl_X Z := \{((a_1, \cdots, a_m), [b_1, \cdots, b_m]) \in \mathbb{A}^m \times \mathbb{P}^{m-1} | a_i b_j = a_j b_i, \forall i, j\}.$$

with a natural projection morphism

$$pr_f : Bl_X Z \to Z, \quad ((a_1, \cdots, a_m), [b_1, \cdots, b_m]) \to (a_1, \cdots, a_m)$$

Then $E_X Z := O \times \mathbb{P}^{n-1}$ is a codimension 1 smooth subvariety of $Bl_X Z$, and pr_f is an isomorphism outside of $E_X Z$. The exceptional divisor $E_X Z$ induces a line bundle $\mathcal{O}_{Bl_X Z}(E_X Z)$ on $Bl_X Z$, and for all integers $i \geq 0$ and n, the cohomology group

$$H^i(Bl_X Z, \mathcal{O}_{Bl_X Z}(-nE_X Z))$$

is an $\mathcal{O}_Z$-module by the push-forward map. When $n \geq -m+1$, we have the following well-known vanishing formula from Kahler geometry

$$H^i(Bl_X Z, \mathcal{O}_{Bl_X Z}(-nE_X Z)) \cong \begin{cases} \mathcal{I}^n, & i = 0 \\ 0, & i > 0, \end{cases} \tag{1.1}$$

where $\mathcal{I}$ is the ideal sheaf of X in Z and we abuse the notation to denote $\mathcal{I}^n := \mathcal{O}_Z$ when $n \leq 0$.

The modern formulation of (1.1) was given by Grothendieck through two generalizations: first, he defined the blow-up as the projective spectrum of the Rees algebra

$$Bl_X Z := \mathrm{Proj}_Z(\bigoplus_{n\geq 0} \mathcal{I}^n).$$

Second, he introduced the derived category of coherent sheaves and derived functors in order to generalize the Serre duality theorem. Using the language of derived categories and noticing $m = codim_X Z$, (1.1) can be reformulated as the canonical equivalence for $n \geq -codim_X Z + 1$:

$$\psi_n : \mathcal{I}^n \to Rpr_{f*}(\mathcal{O}_{Bl_X Z}(-nE_X Z)) \tag{1.2}$$

The advantage of the sheaf formulation (1.2) is that it can be generalized to the case that X and Z are smooth varieties or algebraic stacks. However, in [43], we found a more surprising fact that (1.2) even holds for quasi-smooth derived schemes/stacks, by modifying the definition of Rees algebras in the sense of Hekking [12, 13]

Theorem 1.1 (Zhao [43], generalized vanishing theorem)**.** *The canonical equivalence ψ_n also holds when X and Z are both quasi-smooth derived algebraic stacks if we consider the derived Rees algebra and and the virtual codimension.*

To explain the statement of Theorem 1.1, we first explain the meaning of "quasi-smoothness" and "derived Rees algebra/blow-ups" in Section 1.2 and Section 1.3. Then we will explain how it unified different mathematical phenomenons from different fields.

1.2. **Quasi-smooth derived schemes/stacks.** Derived algebraic geometry is naturally introduced to the setting due to its compatibility with the base change: given a fiber product of algebraic varieties:

$$\begin{array}{ccc} X \times_Z Y & \xrightarrow{pr_2} & Y \\ \downarrow{\scriptstyle pr_1} & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{f} & Z, \end{array}$$

the two morphisms in the cohomology (also algebraic K-theory and derived category)

$$g^* f_* \text{ and } pr_{2*} pr_1^* : H^*(X) \to H^*(Y)$$

are usually different unless the intersection is transversal (i.e. the sheaf $tor^i_Y(\mathcal{O}_X, \mathcal{O}_X)$ does not vanish when $i > 0$). This issue can be solved by introducing the derived algebraic geometry and deriving the fiber product $X \times^{\mathbb{L}}_Y X$ which includes enough information about all $tor^i_Y(\mathcal{O}_X, \mathcal{O}_X)$.

A typical example of a quasi-smooth derived scheme is the derived fiber product $X \times^{\mathbb{L}}_Z Y$ where all X, Y, Z are smooth. Its virtual dimension can be defined as $dim(X) + dim(Y) - dim(Z)$. In general, a quasi-smooth derived scheme (Zariski) locally is always such a derived fiber product $X \times_Z Y$ of smooth varieties, and the virtual dimension is a local constant. A quasi-smooth derived stack can be defined as a smooth quotient of a quasi-smooth derived scheme.

Other than the fiber product of smooth varieties, typical examples of quasi-smooth derived stacks also include

(1) The moduli space of stable maps from algebraic curves (with marked points) to a proper smooth variety;

(2) The moduli space of coherent sheaves on an algebraic surface.

1.3. **Derived Blow-ups.** The appearance of derived blow-ups can be illustrated through a simple example: let C_1 and C_2 be two complex smooth algebraic curves in a smooth algebraic surface S. One of the most classical algebraic geometry questions is to define the multiplicity of the intersection and count the intersection numbers of $C_1 \cap_S C_2$. Here are two classical approaches to define it:

(1) The first approach is homological, i.e. we compute the dimension of

$$\sum_{i=0}^{\infty}(-1)^i dim(Tor^i_{\mathcal{O}_p}(\mathcal{O}_{C_1}, \mathcal{O}_{C_2}))$$

where all $Tor^i_{\mathcal{O}_p}(\mathcal{O}_{C_1}, \mathcal{O}_{C_2})$ are finite dimensional vector spaces over $\mathbb{C}$.

(2) The second approach is birational: we consider $\tilde{S} := Bl_p S$. Then $Bl_p C_1 \cong C_1$ and $Bl_p C_2 \cong C_2$ have a canonical lifting to $\tilde{S}$, which we denote as $\tilde{C}_1$ and $\tilde{C}_2$. Then the multiplicity of $\tilde{C}_1 \cap_{\tilde{S}} \tilde{C}_2$ is the multiplicity of $C_1 \cap_S C_2$ minus 1. We keep this procedure until those two curves no longer intersect, and the multiplicity of $C_1 \cap_S C_2$ is the number of points we blow up consecutively.

The above two approaches coincide when both C_1 and C_2 are smooth but will give different answers if C_1 or C_2 is singular: let S be the affine plane with coordinates x, y and C_1, C_2 be the zero locus of xy and $x+y$ respectively. Then two curves C_1 and C_2 have homological intersection multiplicity 2 at the origin point O, but no longer intersect after blowing up.

To modify the second approach, we define $\tilde{C}_1 := Bl_O C_1 \cup \mathbb{P}^1$ and $\tilde{C}_2 := Bl_O C_2$, where $\mathbb{P}^1$ is the exceptional divisor of $Bl_O S$. Then $\tilde{C}_1$ and $\tilde{C}_2$ intersect transversally with multiplicity 1. Moreover, we will notice that $\tilde{C}_1$ only depends on the origin point and C_1 but not the close embedding to $\mathbb{A}^2$, and thus is intrinsic. This modification was first observed by Kiem-Li [22], which they called as the "intrinsic blow-up". Nowadays, we know that the intrinsic blow-up is the shadow of derived blow-ups on the classical schemes.

The main ingredient of the derived blow-up is a derived enhancement of the formal neighborhood. Given a closed embedding of schemes $f : X \to Z$ such that $\mathcal{I}$ is the ideal sheaf, the classical blow-up is defined as

$$Bl_X Z := \mathrm{Proj}_Z \bigoplus_{n\in\mathbb{Z}_{>0}} \mathcal{I}^n.$$

When X or Z are not smooth, $\mathcal{I}^n/\mathcal{I}^{n+1}$ and $Sym^n_Z(\mathcal{I}/\mathcal{I}^2)$ in general are not the same, where the latter (or more precisely $Sym^n_Z(C_f)$ where C_f is the conormal complex) behaves better in the intersection theory. Thus the modification is obtained by modifying $\mathcal{I}^n$ to construct an $\mathbb{Z}$-graded $\mathcal{O}_X[t^{-1}]$-algebra

$$R^{ext}_{X/Z} := \bigoplus_{d\in\mathbb{Z}} (R^{ext}_{X/Z})_d$$

such that

(1) $(R^{ext}_{X/Z})_n \cong \mathcal{O}_Z$ if $n \le 0$ and the underlying scheme of $(\mathcal{O}_Z/R^{ext}_{X/Z})_n$ is the n-th formal neighborhood of X in Z;

(2) we have the equation

$$(R^{ext}_{X/Z})_n/(R^{ext}_{X/Z})_{n+1} \cong f_* Sym^n_Z(C_f). \tag{1.3}$$

where C_f is the cotangent complex of f (which is a generalization of the cotangent bundle to the case that f is not a regular embedding).

(3) $(R^{ext}_{X/Z})_n \cong \mathcal{I}^n$ when X and Z are both smooth varieties.

The above modification was accomplished by Hekking [12], while part of the construction had been considered by Gaitsgory-Rozenblyum [9] and Halpern-Leistner [11]. Hekking [12] defined the derived blow-up $\mathbb{B}l_X Z$ just as

$$\mathbb{B}l_X Z := \mathrm{Proj}_Z \bigoplus_{n\in\mathbb{Z}_{\geq 0}} (R^{ext}_{X/Z})_d$$

1.4. **K-theoretic consequences of the generalized vanishing theorem.** At the level of Grothendieck group of coherent sheaves, Theorem 1.1 induces the K-theoretic comparison formula

Theorem 1.2 (Zhao [43])**.** *Let C_f be the conormal complex of X in Z. Then*

$$\begin{aligned}[\mathcal{O}_Z] &= pr_{f*}[\mathcal{O}_{\mathbb{B}l_X Z}(-vcodim_X Z+1)] + f_*(\sum_{j=0}^{-vcodim_X Z}[Sym^i_X(C_f)]) \\ &= pr_{f*}([\mathcal{O}_{\mathbb{B}l_X Z}]) + (-1)^{vcodim_X Z} f_*(\sum_{l=0}^{-vcodim_X Z} det(C_f)^{-1}[Sym^l_X(C_f)^\vee)],\end{aligned}$$

Excess intersection formula. Thomas's excess intersection formula (Theorem 3.1 of [35]) is a direct corollary of Theorem 1.2: a (non-derived) Cartesian diagram of smooth varieties

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow g & & \downarrow \\ X' & \xrightarrow{f'} & Y', \end{array}$$

with a short exact sequence of locally free sheaves $0 \to N \to g^*C_{f'} \to C_f \to 0$ on X induces a canonical morphism $r : X \to Y \times^{\mathbb{L}}_{Y'} X'$, where $\mathbb{L}$ means the derived fiber product.

Theorem 1.3 (Zhao)**.** *The cotangent complex of r is $N[-1]$ and the derived blow-up of X in $Y \times_{Y'} X'$ is the empty scheme. Moreover, by Theorem 1.2, we have*

$$[\mathcal{O}_{Y\times_{Y'}X'}] = r_*(\sum_{i\geq 0}(-1)^i[\wedge^i N]),$$

This is Thomason's excess intersection formula (Theorem 3.1 of [35]*).*

Virtual fundamental class. The quasi-smooth derived algebraic stack is closed related to the enumerative geometry due to the following fact: given a quasi-smooth derived scheme $\mathcal{X}$, the class of structure sheaf $[\mathcal{O}_\mathcal{X}]$ is the K-theoretic virtual fundamental class in the sense of Li-Tian [25], Beherend-Fantechi [2] (precisely speaking, we need to restrict to the underlying classical stack). If we endow $\mathcal{X}$ with a torus T action, the fixed loci $\mathcal{X}^T$ is also quasi-smooth. Let $f : X^T \to X$ and $\tau : E_X Z \to \mathbb{B}l_X Z$ be the canonical closed embedding. If $vdim(\mathcal{X}^T) < vdim(\mathcal{X})$, by Theorem 1.2 we induce formula

$$[\mathcal{O}_{\mathcal{X}}] = pr_{f*}[\mathcal{O}_{\mathbb{B}l_{\mathcal{X}^T}\mathcal{X}}] = pr_{f*}\tau_*(\frac{1}{1-[C_\tau]}) = pr_{f*}\tau_*(\sum_{i\geq 0}[C_\tau^i]) \tag{1.4}$$
$$= f_*(\sum_{i\geq 0}[Sym^i(C_f)]) = f_*(\frac{1}{1-[C_f]}),$$

which is the virtual localization formula (equation (1) of Graber-Pandharipande [10]). The rigorous proof and the case $vdim(\mathcal{X}^T) \geq vdim(\mathcal{X})$ are given in [43]:

Theorem 1.4 (Zhao, [43])**.** *The equality $[\mathcal{O}_{\mathcal{X}}] = f_*(\frac{1}{1-[C_f]})$ always holds in the localized Grothendieck group when X is quasi-smooth.*

1.5. **Hecke correspondences and resolution of the diagonal.** Hecke correspondence is a classical construction of algebras and representations in the geometric representation theory: let

$$f : X \to Y$$

be a morphism of algebraic varieties (or stacks), and let

$$\pi_{12}, \pi_{13}, \pi_{23} : X \times_Y X \times_Y X \to X \times_Y X$$

be the three natural projection morphisms. Then the morphism

$$(F, G) \to \pi_{13*}(\pi_{12}^* F \otimes \pi_{23}^* G)$$

induces a convolution algebra structure on the cohomology group (resp. algebraic K-theory and derived categories) of $X \times_Y X$ which has a canonical representation on $H^*(X)$ (resp. $K(X)$ and $D^b(X)$). Here we give three typical examples of K-theoretic Hecke correspondences:

(1) the affine Hecke algebra as the convolution algebra of Steiberg variety (Chriss-Ginzburg [7]);
(2) the quantum loop group representations through the Hecke correspondences of quiver varieties (Nakajima [27, 28]);
(3) the Hecke correspondence for smooth moduli space of sheaves on algebraic surfaces (Neguţ [29, 30])

The derived algebraic geometry is naturally introduced in the convolution algebra, as its associativity requires the base change formula

$$f^* f_* = pr_{2*} pr_1^* : H^*(X) \to H^*(X)$$

where $pr_1, pr_2 : X \times_Y X \to X$ are the canonical projections. Moreover, at the level of categories, Ben-Zvi, Francis and Nadler [4] proved that there is a canonical equivalence between Fourier-Mukai kernels and derived functors

$$QCoh(X \times_Y X') \cong Fun_{QCoh(Y)}(QCoh(X), QCoh(X'))$$

when $f : X \to Y$ is a perfect morphism to a derived stack Y with affine diagonal.

Let $\Delta_f : X \to X \times_Y^{\mathbb{L}} X$ be the diagonal morphism. At the level of algebraic K-theory or derived category, two objects play a crucial role in the study of the convolution algebra: $\Delta_{f*}\mathcal{O}_X$ which is the identity element of $K(Y \times_X^{\mathbb{L}} Y)$ and $\mathcal{O}_{X \times_Y^{\mathbb{L}} X}$ which represents the functor $f^* f_*$. In [43, 44], we studied the Hecke correspondences when f is the natural projection morphism of (stacky) blow-ups of smooth varieties:

Theorem 1.5. *Let $f: X \to Z$ be a regular embedding of smooth varieties. Then $Bl_X Z$ is the derived blow-up of $E_X Z \times_X E_X Z$ in $Bl_X Z \times^{\mathbb{L}}_Z Bl_X Z$, and the diagonal embedding $\Delta_f : Bl_X Z \to Bl_X Z \times^{\mathbb{L}}_Z Bl_X Z$ is the projection morphism and $E_X Z$ is (virtual) exceptional divisor. Here $\mathbb{L}$ means the derived fiber product in the sense of Lurie* [26] *and Toen-Vezzosi* [37].

At the level of derived categories, there is a two-term complex of locally free sheaves Bel on $E_X Z \times_X E_X Z$ generated by the tautological line bundles and (relative) cotangent bundles, and $f_i \in D^b_{coh}(Bl_Z X \times^{\mathbb{L}}_Z Bl_Z X)$ for $i \in \mathbb{Z}_{\geq 0}$ such that

(1) $Cone(f_{i+1} \to f_i) \cong R\gamma_*(Sym^i(Bel))$, where $\gamma : E_X Z \times_X E_X Z \to Bl_X Z \times^{\mathbb{L}}_Z Bl_X Z$ is the canonical closed embedding;
(2) $Sym^i(Bel) \cong R\Delta_* \mathcal{O}_{E_X Z}(i)$ when $i \geq codim_Z X$, where Δ is the diagonal embedding of $E_X Z$ and $\mathcal{O}_{E_X Z}(1)$ is the tautological bundle,
(3) $f_0 \cong \mathcal{O}_{Bl_X Z \times^{\mathbb{L}}_Z Bl_X Z}$ and $f_n \cong R\Delta_{f*}\mathcal{O}_{Bl_X Z}(-nE_Z X)$ when $n \geq codim_Z X - 1$.

We remark that Orlov's semi-orthogonal decomposition theorem for blow-up of smooth varieties [32] *is a direct result of Theorem 1.5.*

Theorem 1.6 (Zhao [44])**.** *For any algebraic stack $\mathcal{X}$ with an effective divisor Cartier divisor D and a positive integer l, let $\mathcal{X}_{D,l}$ be the l-th root stack defined by Cadman* [5] *and Abaramovich-Graber-Vistoli* [1]*. Then $D^b_{coh}(\mathcal{X}_{D,l})$ is a canonical representation of the convolution algebra*

$$D^b_{coh}([\mathbb{A}^1/\mathbb{G}^m] \times_{\theta_l, [\mathbb{A}^1/\mathbb{G}^m], \theta_l} [\mathbb{A}^1/\mathbb{G}^m]),$$

where the morphism $\theta_l : [\mathbb{A}^1/\mathbb{G}^m] \to [\mathbb{A}^1/\mathbb{G}^m]$ is the l-th power on both $\mathbb{A}^1$ and $\mathbb{G}_m$. Moreover, the representation induces a canonical basis as the semi-orthogonal decomposition

$$D^b_{coh}(\mathcal{X}_{D,l}) \cong < D^b_{coh}(X), D^b_{coh}(D)(i) >_{0 \leq i \leq i-1} .$$

1.6. **The Birational Geometry of The Nested Quiver Varieties.** Finally we review the Hecke correspondences of Nakajima [27, 28] and Neguţ [29, 30]. Now we come to our original representation-theoretic motivation: the geometry of nested quiver varieties and the categorification of quantum loop groups. Let $\mathfrak{g}$ be a Kac-Moody algebra, we denote $U_q(L\mathfrak{g})$ as the quantum loop algebra, which is a q-deformation of the loop algebra $L\mathfrak{g}$ following Drinfeld's new representation.

Nakajima [27][28] revealed the geometric nature of the quantum loop algebra representations: for any directed graph (which we also call as a quiver) $Q = (V, E)$, Nakajima constructed a quiver variety $\mathcal{M}_Q(v, w)$ for any two dimension vectors $v, w \in \mathbb{Z}^I_{\geq 0}$ such that $w \neq 0$. When Q is the Dynkin diagram of the Kac-Moody algebra $\mathfrak{g}$, Nakajima constructed an action

$$U_q(L\mathfrak{g}) \curvearrowright \bigoplus_{v \in \mathbb{Z}^V_{\geq 0}} K^{\mathbb{G}_m}(\mathcal{M}_Q(v, w))$$

where the q-parameter appears through the $\mathbb{G}_m$-equivariant algebraic K-theory. When Q is the Jordan quiver, v and w are two integers, and

$$\mathcal{M}_Q(v, w)$$

is the moduli space of framed sheaves on the projective plane $\mathbb{P}^2$. Particularly, when $w = 1$, we have $\mathcal{M}_Q(v, 1) \cong (\mathbb{A}^2)^{[v]}$ as the Hilbert scheme of v points on the affine plane $\mathbb{A}^2$. Schiffmann-Vasserot [33] and Feigin-Tsymbaliuk [8] constructed the Fock representation of the quantum toroidal algebra

$$U_{q_1,q_2}(\ddot{gl}_1) \curvearrowright \bigoplus_{v\in\mathbb{Z}_{\geq 0}} K^{\mathbb{G}_m\times\mathbb{G}_m}(\mathcal{M}_Q(v,w)),$$

which was generalized by Neguţ [29, 30] to the moduli space of stable sheaves on algebraic surfaces.

The above action was constructed by the Hecke correspondence, which we take $(\mathbb{A}^2)^{[n]}$ as an example to explain it. We consider the nested Hilbert scheme

$$(\mathbb{A}^2)^{[n,n+1]} := \{(\mathcal{I}_n,\mathcal{I}_{n+1},x)\in(\mathbb{A}^2)^{[n]}\times(\mathbb{A}^2)^{[n+1]}\times\mathbb{A}^2|\mathcal{I}_n/\mathcal{I}_{n+1}\cong\mathbb{C}_x\}$$

with the projections p_n, q_n, π_n to $(\mathbb{A}^2)^{[n]}$, $(\mathbb{A}^2)^{[n+1]}$ and $\mathbb{A}^2$ respectively. The quotient of ideal sheaves induces a tautological line bundle $\mathcal{L}$ on $(\mathbb{A}^2)^{[n,n+1]}$. Then the quantum toroidal algebra action $U_q(\ddot{gl}_1)$ is induced by the following operators

$$e_i := (q_n\times\pi_n)_*\circ(-\otimes\mathcal{L}^i)\circ p_n^*,\quad K^{\mathbb{G}_m\times\mathbb{G}_m}((\mathbb{A}^2)^{[n]})\to K^{\mathbb{G}_m\times\mathbb{G}_m}((\mathbb{A}^2)^{[n+1]}\times\mathbb{A}^2)$$

$$f_i := (p_n\times\pi_n)_*\circ(-\otimes\mathcal{L}^i)\circ q_n^*,\quad K^{\mathbb{G}_m\times\mathbb{G}_m}((\mathbb{A}^2)^{[n+1]})\to K^{\mathbb{G}_m\times\mathbb{G}_m}((\mathbb{A}^2)^{[n]}\times\mathbb{A}^2).$$

One of the key points of Neguţ's computation for the relation of the above operators (for arbitrary surface) is the construction of the following smooth quadruple moduli space $\mathfrak{Y}_n$

$$\{(\mathcal{I}_{n+1}\subset\mathcal{I}_n,\mathcal{I}'_n,\subset\mathcal{I}_{n-1})|\mathcal{I}_n/\mathcal{I}_{n+1}\cong\mathcal{I}_{n-1}/\mathcal{I}'_n\cong\mathbb{C}_x,\mathcal{I}'_n/\mathcal{I}_{n+1}\cong\mathcal{I}_{n-1}/\mathcal{I}_n\cong\mathbb{C}_y\}$$

In [41, 40, 42], we revealed the birational geometry nature of $\mathfrak{Y}_n$:

Theorem 1.7 (Zhao [41, 40, 42])**.** *We have the canonical isomorphisms*

$$\mathfrak{Y}_n \cong \mathbb{B}l_{(\mathbb{A}^2)^{[n,n+1]}}((\mathbb{A}^2)^{[n,n+1]}\times_{(\mathbb{A}^2)^{[n+1]}}(\mathbb{A}^2)^{[n,n+1]}) \tag{1.5}$$

$$\cong \mathbb{B}l_{(\mathbb{A}^2)^{[n-1,n]}}((\mathbb{A}^2)^{[n-1,n]}\times_{(\mathbb{A}^2)^{[n-1]}}(\mathbb{A}^2)^{[n-1,n]}) \tag{1.6}$$

where the projection morphisms are induced by the forget functors. Moreover, the Neguţ's quadruple moduli space can be defined for any quiver varieties, which is smooth and we also have isomorphisms similar to (1.5) *and* (1.6)*.*

As a corollary of Theorem 1.7, we obtained a weak categorification of Neguţ's quantum toroidal algebra action in [41, 40].

2. Ongoing and Future Projects

2.1. **The Derived Birational Geometry.** All the above discussions about the derived Rees algebras and derived blow-ups should be considered in a larger framework, which we personally call as "derived birational geometry". Like the classical birational geometry, we can call two derived schemes/algebraic stacks U and V to be "derived birational equivalent" if there is an open subscheme $U'\subset U$ and $V'\subset V$ such that $U'\cong V'$. Thus it is natural to discuss the geometry of derived schemes/stacks under the derived birational equivalence, especially for those with extra structures (like quasi-smooth or shifted symplectic stacks).

Particularly, one of the central objects in classical birational geometry is the section ring of a line bundle L on an algebraic variety X, which is the commutative ring

$$R(X,L) := \bigoplus_{n\in\mathbb{Z}^{\geq 0}} \Gamma(X,\mathcal{L}^n).$$

Classical birational geometry cares about the following questions:

(1) Is $R(X,L)$ finitely generated?
(2) If so, what is the dimension of $\mathrm{Spec}(R(X,L))$?
(3) How do we compare the geometry of $\mathrm{Proj}(R(X,L))$ or $\mathrm{Spec}(R(X,L))$ with the geometry of X?

The section ring can also be defined relevantly. Similar to the classical birational geometry, we can consider the derived section ring, which is a simplicial ring

$$\mathbb{R}(X,L) := \bigoplus_{n\in\mathbb{Z}^{\geq 0}} R\Gamma(X,\mathcal{L}^n).$$

and ask the following questions:

(1) Is $\mathrm{Spec}(\mathbb{R}(X,L))$ finite type?
(2) If so, what is the virtual dimension of the spectrum?
(3) What is the cotangent complex of $\mathrm{Proj}(\mathbb{R}(X,L))$ or $\mathrm{Spec}(\mathbb{R}(X,L))$ respectively? Particularly, are they still quasi-smooth or shifted symplectic if X is quasi-smooth or shifted symplectic?

Under the framework of derived birational geometry, the generalized vanishing theorem can be reformulated as

Theorem 2.1. *The canonical morphism*

$$\mathbb{B}l_X Z \to \mathrm{Proj}_Z(\bigoplus_{n\geq 0} pr_{f*}\mathcal{O}_{\mathbb{B}l_X Z}(-nE_X Z))$$

is an isomorphism.

Other than the derived blow-up of quasi-smooth schemes/stacks, another typical example in the derived birational geometry is the projectivization of a two-term locally free complex (more precisely, a quasi-coherent sheaf with Tor amplitude $[-1,0]$) V over a derived stack X, which was studied by Jiang [18]. As a corollary of Jiang's generalized Serre theorem [18], we also have

Theorem 2.2. *Let $p_V : \mathbb{P}_X(V) \to X$ be the projection morphism and $\mathcal{O}_{\mathbb{P}_X(V)}(1)$ be the tautological line bundle on $\mathbb{P}_X(V)$, then the canonical morphism*

$$\mathbb{P}_X(V) \to \mathrm{Proj}_X(\bigoplus_{n\geq 0} p_{V*}\mathcal{O}_{\mathbb{P}_X(V)}(n))$$

is an isomorphism.

As the first step toward the derived birational geometry, in [43] we discussed the desingularization of quasi-smooth derived schemes, and proved a desingularization theorem similar to Hironaka [14]:

Theorem 2.3 (Derived Desingularization Theorem)**.** *Given a quasi-smooth derived scheme X which admits a closed embedding into a smooth variety, there exists $f_i : Z_i \subset X_i$ for $0 \geq i \geq n$ such that $X_n \cong \emptyset$, $X_0 \cong X$ and all X_i are the derived blow-up of X_{i-1} along smooth varieties Z_{i-1}.*

Relations with curves counting. A special case of Theorem 2.3 is the desingularization of the moduli space of stable maps of genus 1 curves to $\mathbb{P}^n$ by Vakil-Zinger [38]. The following theorem explains how to induce the Gromov-Witten invariant from the desingularization process:

Theorem 2.4 (Approximation Theorem)**.** *Assuming the setting of Theorem 2.3, let $p_i : Z_i \to X$ be the projection morphism, $\mathcal{F}_i$ be the co-normal complex of Z_i in X_i, r_i be the (virtual) codimension of Z_i in X_i, then*

$$[\mathcal{O}_X] = \sum_{i=1}^{n} (-1)^{r_i} p_{i*}(det(\mathcal{F}_i)^{-1}[\sum_{j=0}^{-r_i} Sym^j(\mathcal{F}_i^{\vee})]). \tag{2.1}$$

2.2. **Categorification of the quantum loop/toroidal/shifted algebra actions.** Another long-term project we will consider is the categorification of the quantum loop/toroidal/shifted algebra and its actions.

2.2.1. *The case of quiver varieties.* As we stated in Section 1.6, our original motivation is to study the categorification of quantum loop algebra actions and the derived category of Nakajima quiver varieties. A direct corollary of Theorem 1.7 is a weak categorification of the commutator of the positive part and negative part of the quantum loop algebra actions.

Let $Q = (V, E)$ be an arbitrary finite quiver. Neguţ, Sala and Schiffmann [31] gave a definition of the quantum loop algebra for a maximal set of deformation parameters, which is generated by elements in $V \times \mathbb{Z}$, modulo explicit quadratic and cubic relations. Its positive part is isomorphic to the localized K-theoretic Hall algebra (in the sense of Konsevich-Soibelmann [23] and Schiffmann-Vasserot [34]) of preprojective representations with no potentials (and thus coincides with the quantum toroidal algebra for the Jordan quiver case). Combined with Theorem 1.7, in [39] we will give a weak categorification of Neguţ-Sala-Schiffmann's quantum loop group action on the equivariant K-theory of Nakajima quiver varieties.

The strong categorification, which is our ultimate goal, needs more attention as it is closely related to the L_∞ algebroid of formal neighborhoods, in the sense of Kapranov [21] and Calaque-Caldurariu-Tu [6]. We discuss the first-order obstruction in [41], and the higher-order obstructions will be studied in our future work.

2.2.2. *The case of derived projectivizations and Grassmanians.* On the other hand, let X be a derived stack and F be a quasi-coherent sheaf over X with tor amplitude $[-1, 0]$. Jiang [18, 19] developed the theory of derived projectivization and derived Grassmanians, and generalized the Borei-Weil-Bott theorem. Recently, Jiang [19] constructed the semi-orthogonal decomposition of for the derived category of derived Grassmanians (the case that X is smooth and F is a two-term complex of locally free sheaves had been proved by Toda [36]).

From the representation theory side, the categorical action of the quantum shifted algebra on its derived category is studied by Hsu [15, 16], who constructed the categorical action of the quantum shifted algebra action on the derived categories of derived Grassmanians. However, this action is not enough to induce the semi-orthogonal decomposition of Toda and Jiang.

In future work, we will give a modification of Hsu's categorical actions such that the semi-orthogonal decomposition of Toda and Jiang forms a canonical basis. The key point is that, we should consider a Drinfeld double of the quantum shifted algebra, where the positive part acts on the derived Grassmanians of F and the negative part acts on the derived Grassmanians of $F^\vee[-1]$ instead.

For the first step, we will give a generalization of Beilinson resolution: let $\Delta_F : \mathbb{P}_X(F) \to \mathbb{P}_X(F) \times_X \mathbb{P}_X(F)$ be the diagonal embedding. When F is a locally free sheaf, Beilinson [3] induced a resolution of diagonal by showing

$$\Delta_{F*}(\mathcal{O}_{\mathbb{P}_X(F)}(l)) \cong Sym^l(Bel), \quad l \geq rank(F), \tag{2.2}$$

where the complex Bel is introduced in Theorem 1.5. The equation (2.2) will no longer hold when F is not locally free. However, through the derived birational geometry, in the future paper we will show that the cone

$$Cone(Sym^l(Bel) \to \Delta_{F*}(\mathcal{O}_{\mathbb{P}_X(F)}(l)))$$

can be computed and related to another derived projectvization $\mathbb{P}_X(F^\vee[-1])$.

Relations with the quantum cohomology. We refer to Kuznetsov [24] for the relation between the derived category of coherent sheaves and quantum cohomology of algebraic varieties. Recently, Iritani [17] and Iritani-Koto [17] studied the quantum cohomology of projective bundles and blow-ups of smooth algebraic varieties. Using our techniques, we expect to generalize the categorical representations to the quantum cohomology of flag varieties of two-term complexes. Moreover, it should verify some special cases of Ruan's cohomological crepant conjecture.

References

[1] Dan Abramovich, Tom Graber, and Angelo Vistoli. Gromov-Witten theory of Deligne-Mumford stacks. *Amer. J. Math.*, 130(5):1337–1398, 2008.
[2] K. Behrend and B. Fantechi. The intrinsic normal cone. *Invent. Math.*, 128(1):45–88, 1997.
[3] Alexander A Beilinson. Coherent sheaves on $\mathbb{P}^n$ and problems of linear algebra. *Functional Analysis and Its Applications*, 12(3):214–216, 1978.
[4] David Ben-Zvi, John Francis, and David Nadler. Integral transforms and Drinfeld centers in derived algebraic geometry. *J. Amer. Math. Soc.*, 23(4):909–966, 2010.
[5] Charles Cadman. Using stacks to impose tangency conditions on curves. *Amer. J. Math.*, 129(2):405–427, 2007.
[6] Damien Calaque, Andrei Căldăraru, and Junwu Tu. On the Lie algebroid of a derived self-intersection. *Adv. Math.*, 262:751–783, 2014.
[7] Neil Chriss and Victor Ginzburg. *Representation theory and complex geometry*. Modern Birkhäuser Classics. 2010.
[8] Boris Feigin and Alexander Tsymbaliuk. Equivariant K-theory of Hilbert schemes via shuffle algebra. *Kyoto Journal of Mathematics*, 51(4):831–854, 2011.
[9] Dennis Gaitsgory and Nick Rozenblyum. *A study in derived algebraic geometry: Volume I: correspondences and duality*, volume 221. American Mathematical Society, 2019.
[10] T. Graber and R. Pandharipande. Localization of virtual classes. *Invent. Math.*, 135(2):487–518, 1999.
[11] Daniel Halpern-Leistner. Derived θ-stratifications and the D-equivalence conjecture. *arXiv preprint arXiv:2010.01127*, 2020.
[12] Jeroen Hekking. Graded algebras, projective spectra and blow-ups in derived algebraic geometry. *arXiv preprint arXiv:2106.01270*, 2021.
[13] Jeroen Hekking. *Blow-ups and normal bundles in derived algebraic geometry and beyond*. PhD thesis, KTH Royal Institute of Technology, 2022.
[14] Heisuke Hironaka. Resolution of singularities of an algebraic variety over a field of characteristic zero. I, II. *Ann. of Math. (2)* **79** *(1964), 109–203; ibid. (2)*, 79:205–326, 1964.
[15] You-Hung Hsu. A categorical action of the shifted $q = 0$ affine algebra. *arXiv preprint arXiv:2009.03579*, 2020.
[16] You-Hung Hsu. Semiorthogonal decomposition via categorical action. *arXiv preprint arXiv:2108.13008*, 2021.
[17] Hiroshi Iritani. Quantum cohomology of blowups. *arXiv preprint arXiv:2307.13555*, 2023.
[18] Qingyuan Jiang. Derived projectivizations of complexes. *arXiv preprint arXiv:2202.11636*, 2022.
[19] Qingyuan Jiang. Derived categories of derived grassmannians. *arXiv preprint arXiv:2307.02456*, 2023.

[20] Qingyuan Jiang and Naichung Conan Leung. Derived category of projectivization and flops. *arXiv preprint arXiv:1811.12525*, 2018.
[21] Mikhail Kapranov. Rozansky–witten invariants via atiyah classes. *Compositio Mathematica*, 115(1):71–113, 1999.
[22] Young-Hoon Kiem and Jun Li. A wall crossing formula of Donaldson-Thomas invariants without Chern-Simons functional. *Asian J. Math.*, 17(1):63–94, 2013.
[23] Maxim Kontsevich and Yan Soibelman. Cohomological Hall algebra, exponential Hodge structures and motivic Donaldson-Thomas invariants. *Communication in Number Theory and Physics*, 5(arXiv: 1006.2706):231–352, 2010.
[24] Alexander Kuznetsov. Semiorthogonal decompositions in families. *arXiv preprint arXiv:2111.00527*, 2021.
[25] Jun Li and Gang Tian. Virtual moduli cycles and Gromov-Witten invariants of algebraic varieties. *J. Amer. Math. Soc.*, 11(1):119–174, 1998.
[26] Jacob Lurie. Spectral algebraic geometry. available at https://www.math.ias.edu/ lurie/papers/SAG-rootfile.pdf.
[27] Hiraku Nakajima. Quiver varieties and Kac-Moody algebras. *Duke Mathematical Journal*, 91(3):515 – 560, 1998.
[28] Hiraku Nakajima. Quiver varieties and finite dimensional representations of quantum affine algebras. *Journal of the American Mathematical Society*, 14(1):145–238, 2001.
[29] Andrei Neguţ. Shuffle algebras associated to surfaces. *Selecta Mathematica. New Series*, 25(3):57, 2019.
[30] Andrei Neguţ. Hecke correspondences for smooth moduli spaces of sheaves. *Publications mathématiques de l'IHÉS*, 135(1):337–418, Jun 2022.
[31] Andrei Neguţ, Francesco Sala, and Olivier Schiffmann. Shuffle algebras for quivers as quantum groups. *arXiv preprint arXiv:2111.00249*, 2021.
[32] D. O. Orlov. Projective bundles, monoidal transformations, and derived categories of coherent sheaves. *Izv. Ross. Akad. Nauk Ser. Mat.*, 56(4):852–862, 1992.
[33] Olivier Schiffmann and Eric Vasserot. The elliptic Hall algebra and the K-theory of the Hilbert scheme of $\mathbb{A}^2$. *Duke Mathematical Journal*, 162(2):279–366, 02 2013.
[34] Olivier Schiffmann and Eric Vasserot. On cohomological Hall algebras of quivers: Generators. *Journal für die Reine und Angewandte Mathematik*, 760:59–132, 2020.
[35] R. W. Thomason. Les K-groupes d'un schéma éclaté et une formule d'intersection excédentaire. *Invent. Math.*, 112(1):195–215, 1993.
[36] Yukinobu Toda. Derived categories of Quot schemes of locally free quotients via categorified Hall products. *Math. Res. Lett.*, 30(1):239–265, 2023.
[37] Bertrand Toën and Gabriele Vezzosi. *Homotopical Algebraic Geometry II: Geometric Stacks and Applications: Geometric Stacks and Applications*, volume 2. American Mathematical Soc., 2008.
[38] Ravi Vakil and Aleksey Zinger. A desingularization of the main component of the moduli space of genus-one stable maps into $\mathbb{P}^n$. *Geom. Topol.*, 12(1):1–95, 2008.
[39] Yu Zhao. Quiver varieties and categorified commutator of functors. in preparation.
[40] Yu Zhao. Moduli space of sheaves and categorified commutator of functors. *arXiv preprint arXiv:2112.12434*, 2021.
[41] Yu Zhao. A categorical quantum toroidal action on Hilbert schemes. *to appear in JIMJ*, 2022.
[42] Yu Zhao. Derived blow-ups and birational geometry of nested quiver varieties. *arXiv preprint arXiv:2303.01063*, 2023.
[43] Yu Zhao. A generalized vanishing theorem for blow-ups of quasi-smooth stacks. *arXiv preprint arXiv:2306.09672*, 2023.
[44] Yu Zhao. Resolution of the diagonal on the root stacks. *arXiv preprint arXiv:2309.06788*, 2023.

Kavli Institute for the Physics and Mathematics of the Universe (WPI), University of Tokyo, 5-1-5 Kashiwanoha, Kashiwa, 277-8583, Japan.
E-mail address: yu.zhao@ipmu.jp