RESEARCH STATEMENT

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My current study focuses on the birational geometry of derived algebraic stacks, and its relations with enumerative geometry, the derived category of coherent sheaves, and representation theory.

The methodology: intersection theory vs the strict transform. The study of solution counting and smoothing of algebraic equations can be traced back to Issac Newton, which was later developed as the intersection theory and the theory of strict transforms. The two theories have a great compatibility when considering the transversal intersection of smooth varieties, which was broken when one variety is singular or the intersection is not transversal. Here we explain this (in)compatibility through two examples:

Example 0.1. Let C_1 and C_2 be two smooth algebraic curves of an algebraic surface S. We can count the number of points of the intersection $C_1 \cap C_2$ through the strict transforms (i.e. blow-ups): let p be a point of $C_1 \cap C_2$. Let $\tilde{S}$ be the blow-up of S along p and $\tilde{C}_1$ and $\tilde{C}_2$ be the strict transforms of C_1 and C_2 on $\tilde{S}$ respectively. Then the number of points of $\tilde{C}_1 \cap \tilde{C}_2$ is the number of points of $C_1 \cap C_2$ decreasing by 1. Thus we can do strict transforms consecutively until they longer intersect and count the number of points we blow up. The procedure fails when either C_1 or C_2 is singular: let S be the affine plane, C_1, C_2 be the curves defining by the equations $xy = 0$ and $x + y = 0$ respectively and p be the origin point. Then $C_1 \cap C_2 = 2$ but $\tilde{C}_1$ does not intersect with $\tilde{C}_2$.

Example 0.2. More generally, let $f : X \to Z$ Be a closed embedding of smooth varieties, $Bl_X Z$ be the blow-up of Z along X and $pr_f : Bl_X Z \to Z$ be the projection map. Given a smooth closed variety $V \subset Z$, we denote $\tilde{V}$ as the strict transform of V along pr_f. If V and X intersect transversally, then the fundamental class of $\tilde{V}$ in $Bl_X Z$ is the pull-back of the fundamental class of V in Z. It fails when the intersection is not transversal: let X be a point, Z be an algebraic surface and V be a smooth curve passing through X. Then the pull-back of the fundamental class of V in Z does not only include the fundamental class of $\tilde{V}$ but also the exceptional divisor class.

To improve the compatibility of the above two theories, it seems natural to develop more complicated intersection formulas, like Fulton [7]. However, we find out that we can also modify the theory of strict transforms, by considering the intrinsic blow-up of Kiem-Li [19] but through the framework of derived algebraic geometry, especially the recent derived blow-up theory of Hekking [10] which has better functorial properties. It leads to numerous applications in enumerative geometry, the derived category of coherent sheaves and representation theory.

Derived birational geometry. Our main theorem is a generalization of a classical vanishing theorem: for the most simple case, we consider the blow-up of the origin point O in the affine space $\mathbb{A}^m$ ($m \geq 2$)

$$Bl_O\mathbb{A}^m := \{((a_1, \cdots, a_m), [b_1, \cdots, b_m]) \in \mathbb{A}^m \times \mathbb{P}^{m-1} | a_i b_j = a_j b_i, \forall i, j\},$$

which contains a codimension 1 submanifold $E \cong \mathbb{P}^{m-1}$ where all coordinates a_i are 0. By Hartogs's theorem, any meromorphic function on the blow-up with poles in E is the pull-back of a holomorphic function on $\mathbb{A}^m$. Moreover, given an integer n, all the local meromorphic functions with orders of zero $\geq n$ at E (and holomorphic outside of E) form a holomorphic line bundle $\mathcal{O}(-nE)$, whose all higher sheaf cohomology vanishes when $n \geq -m+1$.

From the algebro-geometric side, we can reformulate and generalize the above vanishing theorem for any closed embedding of smooth varieties $f : X \to Z$, using the derived category of coherent sheaves developed by Grothendieck [6]: let $Bl_X Z$ be the blow-up of Z along X, with the exceptional divisor $E_X Z$ and the projection morphism $pr_f : Bl_X Z \to Z$. Let I be the ideal sheaf of X in Z, and I^n be the n-th power of the ideal sheaf I when $n \geq 0$ and be $\mathcal{O}_Z$ when $n < 0$. The vanishing theorem can be reformulated as a canonical isomorphism

$$I^n \cong Rpr_{f*}(\mathcal{O}(-nE_X Z)), \quad n \geq -codim_X Z + 1 \tag{0.1}$$

where Rpr_{f*} is the derived push-forward functor. Our major observation in [32] is that (0.1) can even be generalized to a closed embedding of holomorphic Kuranishi spaces (more precisely, we use the algebro-geometric terminology "quasi-smooth derived algebraic stacks"):

Theorem 0.3. *The equivalence* (0.1) *also holds when both X and Z are quasi-smooth derived algebraic stacks if we consider the derived blow-up of Hekking* [9, 10]*. Moreover, the derived blow-up $\mathbb{B}l_X Z$ (which we use to distinguish with the classical blow-up $Bl_X Z$ and they coincide when both X and Z are smooth) is also quasi-smooth.*

Algebraic K-theory and enumerative geometry. Illusie [12] generalized the conormal bundle to a conormal complex for any closed embedding of quasi-smooth derived stacks, which is locally a morphism of two holomorphic vector bundles (up to quasi-isomorphisms). Its index, i.e. the difference between the dimension of cokernels and kernels at closed points, is a constant on X and we define it as the codimension of the closed embedding, which can be negative. In [32], we studied the algebraic K-theory consequences of Theorem 0.3:

Theorem 0.4 (Zhao [32])**.** *Assuming the setting of Theorem 0.3, let r be the codimension of X in Z, and $G_0(Z)$ be the Grothendieck group of coherent sheaves on Z, i.e. the free abelian group generated by coherent sheaves on Z modulo the short exact sequences. Then in $G_0(Z)$*

$$\begin{aligned}[\mathcal{O}_Z] &= pr_{f*}[\mathcal{O}((-r+1)E_X Z)] + f_*(\sum_{l=0}^{-r}[Sym^l_X(C_f)]) \\ &= pr_{f*}([\mathcal{O}_{\mathbb{B}l_X Z}]) + (-1)^r f_*(\sum_{l=0}^{-r}[det(C_f)^{-1} Sym^l_X(C_f)^\vee)],\end{aligned}$$

where C_f is the conormal bundle of X in Z and Sym, det and $\vee$ denote the (derived) symmetric power, determinant, and dual of a complex.

Excess intersection formula. The excess intersection formula of Thomason (Theorem 3.1 of [24]) is a direct corollary of Theorem 0.4:

Theorem 0.5 (Zhao)**.** *Given a (non-derived) Cartesian diagram of smooth varieties where both f and f' are closed embedding*

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow \\ X' & \xrightarrow{f'} & Y', \end{array}$$

it induces a canonical closed embedding $\theta : X \to Y \times^{\mathbb{L}}_{Y'} X'$, where $\mathbb{L}$ means the derived fiber product and the conormal complex of θ is the restriction morphism of conormal bundles

$$C_{f'}|_X \to C_f.$$

Moreover, if the above restriction morphism of conormal bundles is surjective, we denote N as its kernel. Then the derived blow-up of X in $Y \times^{\mathbb{L}}_{Y'} X'$ is the empty scheme and thus

$$[\mathcal{O}_{Y\times^{\mathbb{L}}_{Y'}X'}] = \theta_*(\sum_{i=0}^{rank(N)} (-1)^i[\wedge^i N]),$$

which is Thomason's excess intersection formula (Theorem 3.1 of [24]*).*

Virtual fundamental class. Given a quasi-smooth derived scheme $\mathcal{X}$, its structure sheaf class is exactly the (K-theoretic) virtual fundamental class in the sense of Li-Tian [21] and Beherend-Fantechi [2] (precisely speaking, we need to restrict to the underlying classical scheme). Moreover, if it has a torus T action, then the fixed locus $\mathcal{X}^T$ is also quasi-smooth. By derived blowing-up the fixed locus and applying Theorem 0.4, we gave a new proof of the (K-theoretic) virtual localization theorem (Thomason [25] and Graber-Pandharipande [8])

Theorem 0.6 (Zhao, [32])**.** *Let $f : \mathcal{X}^T \to \mathcal{X}$ be the canonical closed embedding. The equality $[\mathcal{O}_\mathcal{X}] = f_*(\frac{1}{1-[C_f]})$ always holds in the localized Grothendieck group when $\mathcal{X}$ is a quasi-smooth derived scheme, where C_f is the conormal complex of f.*

Hecke correspondences and derived category of coherent sheaves. The study of kernels and their convolution algebras is a central topic in different fields with very different faces: it is called Fourier transforms in analysis, correspondences in geometry, and bi-modules in algebra. In geometric representation theory, it is called "Hecke correspondences": given a morphism of algebraic stacks $X \to Y$, the three projection morphisms

$$\pi_{12}, \pi_{13}, \pi_{23} : X \times^{\mathbb{L}}_Y X \times^{\mathbb{L}}_Y X \to X \times^{\mathbb{L}}_Y X$$

induces a convolution algebra structure on the cohomology group $H^*(X \times^{\mathbb{L}}_Y X)$ (resp. the Grothendieck group $G_0(X \times^{\mathbb{L}}_Y X)$ or the derived category of coherent sheaves $D^b_{coh}(X \times^{\mathbb{L}}_Y X)$):

$$(F, G) \to \pi_{13*}(\pi_{12}^* F \otimes \pi_{23}^* G)$$

which induces a canonical representation on $H^*(X)$ (resp. $G_0(X)$ or $D^b_{coh}(X)$). In [32, 33], we studied the Hecke correspondences for the natural projection morphisms of (stacky) blow-ups of smooth varieties:

Theorem 0.7. *Let $X \to Z$ be a closed embedding of smooth varieties. Then $Bl_X Z$ is the derived blow-up of the canonical closed embedding*

$$\gamma : E_X Z \times_X E_X Z \to Bl_X Z \times^{\mathbb{L}}_Z Bl_X Z,$$

where the diagonal embedding

$$\Delta_B : Bl_X Z \to Bl_X Z \times^{\mathbb{L}}_Z Bl_X Z, \quad \Delta_E : E_X Z \to E_X Z \times_X E_X Z$$

are the natural projection morphisms of derived blow-up and the exceptional divisor (to the exceptional locus) respectively.

Let π_1, π_2 be the two natural projection morphisms from $E_X Z \times_X E_X Z$ to $E_X Z$. Beilinson [3] *constructed a canonical morphism*

$$Bei : \pi_1^* L_{E_X Z/X} \to \pi_2^*(\mathcal{O}(-E_X Z)|_{E_X Z})$$

where $L_{E_X Z/X}$ is the relative cotangent bundle of $E_X Z$ over X. Then starting from f_0 as the structure sheaf of $Bl_X Z \times^{\mathbb{L}}_Z Bl_X Z$, we can construct $f_{i+1} \in D^b_{coh}(Bl_X Z \times^{\mathbb{L}}_Z Bl_X Z)$ inductively as the mapping cylinder of a canonical morphism $f_i \to R\gamma_*(Sym^i(Bei))$ such that when $i \geq codim_X Z - 1$

$$f_i \cong R\Delta_{B*}\mathcal{O}(-iE_X Z), \quad Sym^i(Bei) \cong R\Delta_{E*}(\mathcal{O}(-iE_X Z)|_{E_X Z}).$$

Regarding $D^b_{coh}(Bl_X Z)$ as a representation of $D^b_{coh}(Bl_X Z \times^{\mathbb{L}}_Z Bl_X Z)$, the above formulas induce a canonical basis as the semi-orthogonal decomposition of Orlov [23].

Theorem 0.8 (Zhao [33])**.** *For any algebraic stack $\mathcal{X}$ with an effective Cartier divisor D and a positive integer l, let $\mathcal{X}_{D,l}$ be the l-th root stack defined by Cadman* [4] *and Abaramovich-Graber-Vistoli* [1]*. Then $D^b_{coh}(\mathcal{X}_{D,l})$ is a canonical representation of the convolution algebra*

$$D^b_{coh}([\mathbb{A}^1/\mathbb{G}_m] \times_{\theta_l, [\mathbb{A}^1/\mathbb{G}_m], \theta_l} [\mathbb{A}^1/\mathbb{G}_m]),$$

where the morphism $\theta_l : [\mathbb{A}^1/\mathbb{G}_m] \to [\mathbb{A}^1/\mathbb{G}_m]$ is the l-th power map on both $\mathbb{A}^1$ and $\mathbb{G}_m$. Moreover, the representation induces a canonical basis as the semi-orthogonal decomposition

$$D^b_{coh}(\mathcal{X}_{\mathcal{D},l}) \cong < D^b_{coh}(\mathcal{X}), D^b_{coh}(\mathcal{D})(i) >_{0 \leq i \leq l-1} .$$

Nakajima quiver varieties and categorical representation theory. In [30, 29, 32], we studied the derived birational geometry of nested Nakajima quiver varieties and nested moduli space of stable sheaves on algebraic surfaces. Here we take $(\mathbb{A}^2)^{[n]}$, the Hilbert schemes of points on the affine plane, as an example: we consider the nested Hilbert scheme

$$(\mathbb{A}^2)^{[n,n+1]} := \{(\mathcal{I}_n, \mathcal{I}_{n+1}, x) \in (\mathbb{A}^2)^{[n]} \times (\mathbb{A}^2)^{[n+1]} \times \mathbb{A}^2 | \mathcal{I}_n/\mathcal{I}_{n+1} \cong \mathbb{C}_x\}.$$

To construct the quantum toroidal algebra action on the Grothendieck groups of Hilbert scheme of points on surfaces (and more general, the moduli space of stable sheaves), Neguţ [22] constructed the following smooth quadruple moduli space $\mathfrak{Y}_n$, which contains quadruples of ideal sheaves:

$$\{(\mathcal{I}_{n+1} \subset \mathcal{I}_n, \mathcal{I}'_n, \subset \mathcal{I}_{n-1}) | \mathcal{I}_n/\mathcal{I}_{n+1} \cong \mathcal{I}_{n-1}/\mathcal{I}'_n \cong \mathbb{C}_x, \mathcal{I}'_n/\mathcal{I}_{n+1} \cong \mathcal{I}_{n-1}/\mathcal{I}_n \cong \mathbb{C}_y\}$$

In [30, 29, 31], we revealed the birational geometry nature of $\mathfrak{Y}_n$:

Theorem 0.9 (Zhao [30, 29, 31]). *The smooth variety $\mathfrak{Y}_n$ is isomorphic to the derived blow-up of both*

$$(\mathbb{A}^2)^{[n,n+1]} \times_{(\mathbb{A}^2)^{[n+1]}} (\mathbb{A}^2)^{[n,n+1]} \text{ and } (\mathbb{A}^2)^{[n-1,n]} \times_{(\mathbb{A}^2)^{[n-1]}} (\mathbb{A}^2)^{[n-1,n]}$$

along the diagonals, where the forgetful functors induce the projection morphisms. Moreover, it can be naturally generalized to any quiver varieties or moduli space of stable sheaves on algebraic surfaces.

As a corollary of Theorem 0.9, we obtained a weak categorification of Neguţ's quantum toroidal algebra action in [30, 29], and will obtain in a weak categorification of the quantum loop algebra action in [28].

Desingularization of Quasi-smooth Derived Schemes. In [32] we discussed the desingularization of quasi-smooth derived schemes, and proved a desingularization theorem similar to Hironaka [11]:

Theorem 0.10 (Derived Desingularization Theorem). *Given a quasi-smooth derived scheme X with a closed embedding into a smooth variety, starting from $X_0 := X$, we can construct X_i inductively as the derived blow-up of X_{i-1} along a smooth center Z_{i-1} such that $X_n \cong \emptyset$ for some n.*

Relations with curves counting. An example of Theorem 0.10 is the desingularization of the moduli space of stable maps of genus 1 curves to $\mathbb{P}^n$ by Vakil-Zinger [27]. The following theorem explains how to induce the Gromov-Witten invariant from the desingularization process:

Theorem 0.11 (Approximation Theorem). *Assuming the setting of Theorem 0.10, let $p_i : Z_i \to X$ be the projection morphism, and $\mathcal{F}_i$ be the conormal complex of Z_i in X_i. Then we have the formula in $G_0(X)$*

$$[\mathcal{O}_X] = \sum_{i=1}^{n} (-1)^{rank(\mathcal{F}_i)} p_{i*}(det(\mathcal{F}_i)^{-1}[\sum_{j=0}^{-rank(\mathcal{F}_i)} Sym^j(\mathcal{F}_i^\vee)]). \tag{0.2}$$

Future Projects. Here we illustrate three projects in different directions:

Categorification of quantum loop/toroidal algebras. Following [30, 29, 31], we could give a weak categorification of quantum loop/toroidal algebra actions on the derived category of quiver varieties. However, to consider the strong categorification, the L_∞-algebroid in the sense of Kapranov [18] and Calaque-Caldararu-Tu [5] has to be considered, which will lead to higher order obstruction theories and we will study in the future work.

Quantum cohomology and derived category of coherent sheaves. The derived category of coherent sheaves is closely related to the quantum cohomology of algebraic varieties (we refer to Kuznetsov [20] as the survey). Recently, Iritani [13] and Iritani-Koto [14] studied the quantum cohomology of projective bundles and blow-ups of smooth algebraic varieties. Using our techniques, we expect to generalize their results to the flag varieties of two-term complexes of holomorphic vector bundles, where the semi-orthogonal decomposition of derived categories of coherent sheaves had been studied by Jiang [17, 15, 16] and Toda [26]. Moreover, it should verify some special cases of Ruan's cohomological crepant conjecture.

Derived birational geometry. Like the classical birational geometry, we define two derived schemes/algebraic stacks to be "derived birational equivalent" if they are isomorphic in an open set, and we are interested in the birational geometry of derived schemes/stacks, especially for the quasi-smooth or shifted symplectic stacks. Particularly, given a line bundle L on a derived scheme X, we consider the simplicial section ring

$$\mathbb{R}(X, L) := \bigoplus_{n \in \mathbb{Z}^{\geq 0}} R\Gamma(X, L^n).$$

and ask the following questions:

(1) Is $\mathrm{Spec}(\mathbb{R}(X, L))$ finite type and what is its (virtual) dimension?
(2) How do we compute the cotangent complex of the affine and projective spectrum of $\mathbb{R}(X, L)$? Moreover, when do those affine and projective spectrum contain quasi-smooth or shifted symplectic structures?

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