Birtional Geometry of Quasi-smooth derived schemes

Yu Zhao

Kavli IPMU, University of Tokyo

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Section 1

The Basic Question

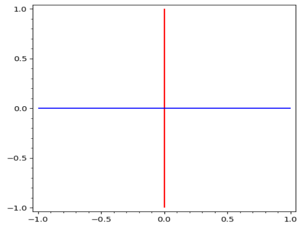
The relation between intersection theory and birational transforms is a basic science question that is understandable by everyone.

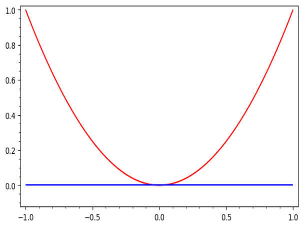
Algebraic geometry and intersection theory

Algebraic geometry studies the solutions of (multi-variable) polynomial equations, which can be transformed into intersection points of figures, with multiplicities counted:

Example

- The x and y-axis have intersection multiplicity one at origin
- The parabola and the tangent line intersect with multiplicity 2





Theorem (Bezout's theorem, first proved by Newton "Principia")

A degree n and degree m curve in the plane intersect at nm points if counting multiplicities and points at infinity.

Cremona transformations And blowing-ups

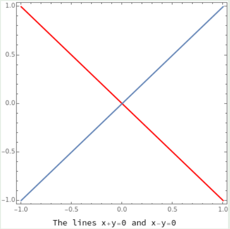
We study the image of curves under the Cremona transformation $Cre(x, y) = (x, y/x)$, which is a bijection outside of the y-axis.

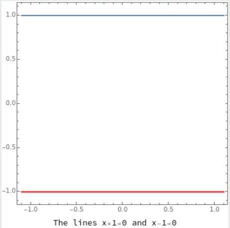
Blowing-ups

For a curve C passing through origin O, we define the image of $C - O$ under the Cremona transformation with all its limit points as the blow-up $Bl_O C$.
The transformation is a bijection if C is smooth at O.

Example

The line $y - ax = 0$ is transformed into $y - a = 0$.

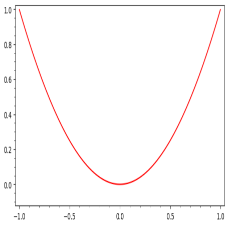


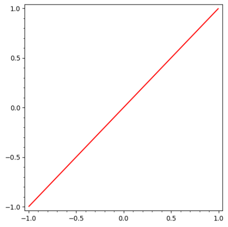


Examples

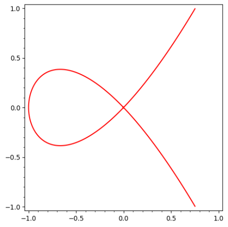
Example

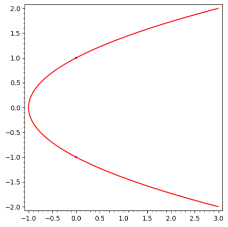
- The parabola $y - x^2 = 0$ is transformed into $y - x = 0$.





- The nodal curve $y^2 - x^3 - x^2 = 0$ is transformed to the smooth parabola $y^2 - x - 1 = 0$.



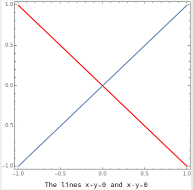


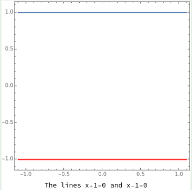
Reducing 1 formula

The multiplicity after the blowing up is always reducing 1 if both two curves are smooth at O.

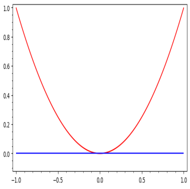
Example

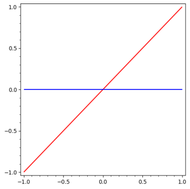
- $x - y = 0$ and $x + y = 0$ no longer intersect after the blowing up.





- The parabola $y - x^2 = 0$ and $y = 0$ have intersection multiplicity 2, which is 1 after the blowing up.



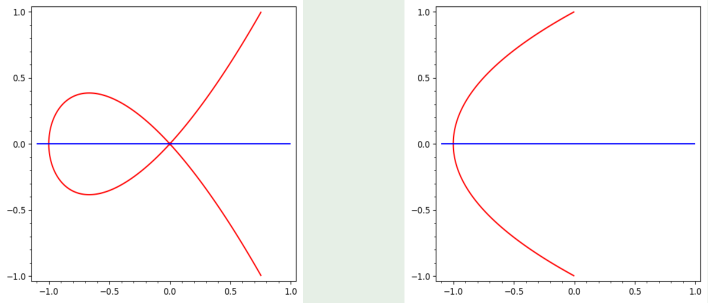


The broken relation in the singular case

It will reduce more if one curve is singular at O, revealing a fundamental incompatibility between birational transformations and intersection theory:

Example

The nodal curve $y^2 - x^3 - x^2 = 0$ intersects with the y-axis as 3 points if counting the intersection multiplicity but only 1 points after the blowing up.

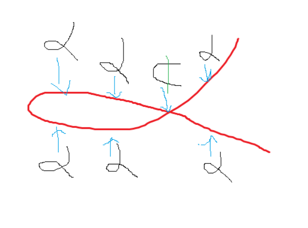


Flatness of blow-ups

Blowing-ups of a family of curves might not be a family, as the intersection number (quantity side) and the number of holes in a curve (shape side) are both constants for families of curves.

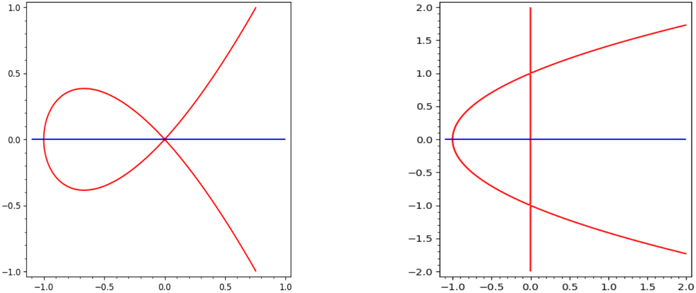
Example

For a family of curves of a nodal curve C with fibers Bl_pC at the point p, the fiber over a smooth point is still C with 1 hole and the fiber over the nodal point is a parabola with no holes. It can be fixed by adding the green line.



The hidden intersection comes from the hidden component

The intersection formula is recovered if we add the line, i.e. y-axis to the blow-up of the nodal curve.



The above example reveals a deep fact:

Other than the classical blow-ups, there are hidden components that store the intersection information. How do we find them?

Section 2

The New Methodology

The solutions require a bit of knowledge, like some abstract algebra, complex analysis, and homology theory. But the more important thing is an open mind and a new methodology.

Derived blow-ups

Hekking's thesis generalized Kiem-Li's "intrinsic blow-up" to any closed embedding of varieties under the framework of **Derived Algebraic Geometry**. I proved that the intersection formula is recovered under the newly derived blow-up theory.

- The derived blow-up shares many common properties, like the exceptional divisor $E_X Z$ for the derived blow-up $\mathbb{R}Bl_X Z$ and $Z - X$ is isomorphic to $\mathbb{R}Bl_X Z - E_X Z$ for $X \subset Z$.
- Classical/derived blow-ups coincide for both smooth $X \subset Z$. In other cases, the derived blow-up might have other "hidden" components that control the intersection theory.
- It relies on the derived structure of varieties.

Derived algebraic geometry vs classical algebraic geometry

Given polynomial equations $g_1, \cdots, g_m \in \mathbb{C}[x_1, \cdots, x_n]$ be polynomial equations, to study the geometry of zero loci:

- Classical algebraic geometry study the quotient ring $\mathbb{C}[x_1, \cdots, x_n]/(g_1, \cdots, g_m)$.
- Derived algebraic geometry consider the $\mathcal{O} := \mathbb{C}[x_1, \cdots, x_n]$ module morphism $\mathcal{O}^m \xrightarrow{\theta_1 := (g_1, \cdots g_m)} \mathcal{O}$ and study the complex

$$\wedge^m \mathcal{O}^m \xrightarrow{\theta_m} \wedge^{m-1}\mathcal{O}^m \to \cdots \to \mathcal{O}^m \xrightarrow{\theta_1} \mathcal{O} \to 0, \qquad (1)$$

where all θ_i are induced by contractions. It is called the derived enhancement of the quotient ring as

$$\mathbb{C}[x_1, \cdots, x_n]/(g_1, \cdots, g_m) \cong coker(\theta_1)$$

- A space locally defined by the complex (1) is called a quasi-smooth derived scheme with dimension $n - m$.

Deformation to normal cone vs (co)normal complex

The classical definition of the blow-up is related to Rees algebra:

- the (extended) Rees algebra $R^{ext}_{X/Z}$ for a closed embedding of algebraic varieties $X \subset Z$ where I is the ideal is defined as

  $$R^{ext}_{X/Z} := \bigoplus_{n \in \mathbb{Z}} I^n.$$

  It is a scheme over $\mathbb{C}$ such that the fiber outside of 0 is Z, and over 0 is the normal cone. Blow-up is its GIT quotient.
- from intersection theory, the (co)normal complex space $\oplus_{n \in \mathbb{Z}} Sym^n(C_{Z/X})$ behaves better than the normal cone, and its deformation was obtained by Gaitsgory-Rozenblyum, Halpern-Leistner, and Hekking.

An example of the derived blow-up

Example (For general audience)

A polynomial $f(x,y)$ such that $f(0,0)=0$ determines a curve C passing through O, and the derived blow-up $\mathbb{R}Bl_O C$ is defined by the polynomial equation

$$\frac{f(x,xy)}{x}=0.$$

Example (For experts)

Given a (local) hypersurface H, the classical blow-up $Bl_H(H\times H)$ of the diagonal $H\subset H\times H$ maps to H by the first projection map. For any $p\in H$, the fiber of the map is the derived blow-up $\mathbb{R}Bl_p H$.

The generalized vanishing (intersection) theorem

The blow-up of the origin point O in the m-dimensional space $\mathbb{C}^m$ ($m \geq 2$) is written by equations

$$\{((a_1, \cdots, a_m), [b_1, \cdots, b_m]) \in \mathbb{C}^m \times \mathbb{P}^{m-1} | a_i b_j = a_j b_i, \forall i, j\},$$

and the exceptional divisor E contains points that all $a_i = 0$.

- By Hartog's theorem, for any integer n there is a bijection between holomorphic function on $\mathbb{C}^m$ with an order of zero $\geq n$, which we denote as I^n, with meromorphic functions on $Bl_O \mathbb{C}^m$ with poles only in E, with order of poles $\leq -n$.
- Moreover, the higher sheaf cohomology $H^i(Bl_O \mathbb{C}^m, \mathcal{O}(-nE))$ vanishes when $n \geq -m + 1$.

The generalized vanishing theorem

Let pr be the projection morphism from $Bl_O\mathbb{C}^m$ to $\mathbb{C}^m$. The vanishing theorem can be rewritten in the language of derived functors

$$Rpr_*\mathcal{O}(-nE)) \cong I^n, \quad -n \geq m+1, \tag{2}$$

where $H^i(Bl_O\mathbb{C}^m, \mathcal{O}(-nE))$ is the cohomology of the complex $Rpr_*\mathcal{O}(-nE))$.

Theorem (Z.)

Equation (2) still holds by replacing O and $\mathbb{C}^m$ with any two quasi-smooth derived schemes under the derived blow-up theory.

Chern characters and the intersection formula

By taking the Chern characters of the above vanishing theorem, we recover the intersection formula.

Section 3

Advanced Applicaitons

The breakthrough of the fundamental question leads to numerous applications in algebraic K-theory, enumerative geometry, derived category of coherent sheaves, and representation theory.

Algebraic K-theory

In algebraic K-theory, the generalized vanishing theorem induces the K-theoretic formula (where r is the codimension)

$$[\mathcal{O}_Z] = pr_{f*}[\mathcal{O}((-r+1)E_X Z)] + f_*(\sum_{l=0}^{-r}[Sym^l_X(C_f)])$$

$$= pr_{f*}([\mathcal{O}_{\mathbb{B}l_X Z}]) + (-1)^r f_*(\sum_{l=0}^{-r}[det(C_f)^{-1}Sym^l_X(C_f)^\vee)],$$

Example (Z.)

The classical fiber product $X \times_Z Y$ can always be embedded to the derived fiber product $X \times^{\mathbb{L}}_Z Y$, and its derived blow-up is empty if all X, Y and Z are smooth and the intersection is "excess". It induces Thomason's excess intersection formula.

Enumerative geometry

For quasi-smooth derived schemes (stacks), the structure sheaf represents the virtual fundamental class in the sense of Li-Tian.

Example (Z.)

Let X be a quasi-smooth derived scheme with a torus T-action. The fixed loci X^T is also quasi-smooth, and the derived blow-up along X^T induces the virtual localization theorem.

Derived category of coherent sheaves

Given a morphism $f : X \to Y$, Fourier-Mukai transforms induces a convolution algebra

$$D^b_{coh}(X \times^{\mathbb{L}}_Y X) \curvearrowright D^b_{coh}(X)$$

- The resolution of the identity element $\mathcal{O}_\Delta \in D^b_{coh}(X \times^{\mathbb{L}}_Y X)$ induces semi-orthogonal decompositions.

Theorem (Z.)

If $X \subset Z$ are both smooth, $Bl_X Z$ is the derived blow-up of

$$E_X Z \times_X E_X Z \subset Bl_X Z \times^{\mathbb{L}}_Z Bl_X Z$$

and the projection morphism is the diagonal embedding.

Orlov's SOD theorem

The following theorem provides a more precise computation than Orlov's SOD theorem.

Theorem (Z.)

There are $f_i \in D^b_{coh}(Bl_X Z \times^{\mathbb{L}}_Z Bl_X Z), i = 0, 1, \cdots$:

- f_{i+1} *is the mapping cylinder of* $f_i \to R\gamma_*(Sym^i(Bei))$
- $f_0 \cong \mathcal{O}_{Bl_X Z \times^{\mathbb{L}}_Z Bl_X Z}$
- *when* $i \geq codim_X Z - 1$

$$f_i \cong R\Delta_{B*}\mathcal{O}(-iE_X Z), \quad Sym^i(Bei) \cong R\Delta_{E*}(\mathcal{O}(-iE_X Z)|_{E_X Z}).$$

Here Bei *is Beilinson's two-term complex.*

Quiver varieties and quantum loop/toroidal algebras

Let $S^{[n]}$ and $S^{[n,n+1]}$ be the Hilbert scheme and nested Hilbert scheme of n (resp. $n, n+1$) points over an algebraic surface S:

Theorem (Z.)

The derived blow-up of diagonals in both

$$S^{[n,n+1]} \times_{S^{[n+1]}} S^{[n,n+1]}, \quad S^{[n-1,n]} \times_{S^{[n-1]}} S^{[n-1,n]}$$

are isomorphic and smooth. It can be generalized to arbitrary Nakajima quiver varieties and induces weak categorifications of quantum toroidal/loop algebras representations.